The Power of a Single Vote

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The purpose of this note is to make the point that under a hypothetical but interesting set of conditions, the probability of a tie vote in an election decreases extremely slowly with the size of the electorate. In fact this probability decreases less than linearly in the size of the electorate. This may not be the fundamental theorem of democracy, but it has to be helpful.

Consider the following situation:

1. We can split our electorate into four groups: Red, Blue, Undecided and You. The Reds will vote red with probability 1. The Blues vote blue with probability 1. The Undecideds are equally likely to go either way.
2. There are as many Reds as Blues.
3. There are an even number of Undecideds.[1]
4. The Undecideds vote independently. No pun intended.
5. If the Undecided vote splits evenly, there are no attempts to change the outcome by recounts, suits, and the like.

Given this unrealistic but still interesting election, what is the probability that the Undecideds split evenly in which case your vote will decide the election?

Suppose the number of Undecided voters is n. Elementary probability tells us that the probability of the Undecideds splitting evenly in this situation is

$$P(evensplitl|n) = \frac{n!}{(n/2)!(n/2)!}0.5^n$$

The problem is that even for a small electorate comprising, say, 5000 Undecideds, we are dealing with astronomically large numbers which are difficult for even a modern computer to compute directly. Fortunately, the probability of an even split obeys an extremely simple recurrence relation.

$$P(evensplitl|n+2) = \frac{n+1}{n+2}P(evensplit|n).$$

See Appendix for the simple proof.

If there are only 2 Undecideds, the probability of an even split is a half, and it is trivial to use the recurrence relation to compute the even split probabilities for 4, 6, 8, and so on. This has been done in Table 1 which shows the probability of an even split as a function of the number of Undecideds.

It is astonishing to see how slowly the probability of an even split decreases. Each doubling of the Undecided population only decreases the probability of an even split by about 0.7. Probability is full of counter-intuitive results. The above very slow fall off of the probability depends critically on the binomial distribution's tendency to put a lot of its probability close to the mean. It's an interesting –and at least for me, totally unexpected – consequence of the so-called Law of Large Numbers.

Of course, this sort of reasoning applies only if the number of Reds is almost the same as the number of Blues, and the Undecided probability is very close to 0.5. But the democratic process tends to work toward such situations, for any political party which seeks a majority has to espouse policies that over-time tend to create this kind of split.

[1] The extension to even or odd number of Undecideds is left as an exercise for the student. The overall conclusion will be the same except that your vote can only force a tie that would not otherwise have occurred.

UNDECIDED VOTERS	PROB OF EVEN SPLIT
2	0.5
4	0.375
8	0.273438
16	0.196381
32	0.13995
64	0.0993468
128	0.0703861
256	0.0498191
512	0.0352446
1024	0.0249278
2048	0.0176288
4096	0.0124662
8192	0.00881519
16384	0.00623338
32768	0.0044077
65536	0.00311672
131072	0.00220386
262144	0.00155837
524288	0.00110193
1048576	0.000779184

Table 1: Probability of a tie vote as a function of the number of undecideds

Appendix

Let n be even and consider the probability of a even split for an electorate comprising $n+2$ Undecideds. If the Undecideds vote independently with a probability of 0.5, according to the binomial density the probability of a tie vote is

$$P(evensplit|n+2) = \frac{(n+2)!}{((n+2)/2)!((n+2)/2)!}0.5^{n+2}$$

$$P(evensplit|n+2) = \frac{(n+2)(n+1)n!}{(n/2+1)(n/2)!(n/2+1)(n/2)!}0.5^2 0.5^n$$

$$P(evensplit|n+2) = \frac{(n+2)(n+1)}{2(n/2+1)2(n/2+1)}\frac{n!}{(n/2)!(n/2)!}0.5^n$$

$$P(evensplit|n+2) = \frac{(n+2)(n+1)}{(n+2)(n+2)}P(evensplit|n)$$

$$P(evensplit|n+2) = \frac{n+1}{n+2}P(evensplit|n)$$

Obviously, as n goes to infinity $P(evensplit|n+2)$ goes to $P(evensplit|n)$.
If $p \neq 0.5$, we get the slightly more complicated recurrence relation

$$P(evensplit|p, n+2) = \frac{(n+2)(n+1)p(1-p)}{(n/2+1)(n/2+1)}P(evensplit|p, n)$$